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Nonequilibrium Thermodynamics and the Transport Phenomena in Magnetically Confined Plasmas

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The neoclassical theory of transport in magnetically confined plasmas is reviewed. The emphasis is laid on a set of relationships existing among the banana transport coefficients. The surface-averaged entropy production in such plasmas is evaluated. It is shown that neoclassical effects emerge from the entropy production due to parallel transport processes. The Pfirsch–Schlüter effect can be clearly interpreted as due to spatial fluctuations of parallel fluxes on a magnetic surface: the corresponding entropy production is the measure of these fluctuations. The banana fluxes can be formulated in a "quasi-thermodynamic" form in which the average entropy production is a bilinear form in the parallel fluxes and the conjugate generalized stresses. A formulation as a quadratic form in the thermodynamic forces is also possible, but leads to anomalies, which are discussed in some detail.

KEY WORDS: Nonequilibrium thermodynamics; entropy production; plasma transport theory; neoclassical transport.

1. INTRODUCTION

Forty years ago appeared Ilya Prigogine's "Thèse d'agrégation," entitled "Etude Thermodynamique des Phénomènes Irréversibles."[1] This work constituted the first systematic account of an emerging science. The basic role played by entropy production in understanding irreversible processes and, in particular, transport phenomena was clearly put forward.

Prigogine's interests subsequently turned to a host of other problems and the subject of linear nonequilibrium thermodynamics was considered as a more or less finished field.

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In this paper, which I dedicate to Ilya Prigogine as a token of gratitude and friendly affection, I wish to show that the concept of entropy production can still play a nontrivial role in the analysis of phenomena that, although expressed in a final form that seems very simple, actually cover a great complexity at the microscopic level.

I choose as an example for this demonstration the theory of transport in magnetically confined plasmas, such as those used in the thermonuclear fusion program. From the point of view of statistical physics, these systems present, as a unique peculiarity, an interplay between the basic transport phenomena and the geometrical characteristics of the system. Let us be clear about this statement. The point is not in the solution of a standard transport equation (say: the diffusion equation) with boundary conditions of unusual (say: toroidal) geometry. It concerns the fact that the mechanisms of the transport themselves (say: the value of the diffusion coefficient) are affected by the global geometry of the magnetic field.

Although these effects have been known for 20 years, their final formulation, as a mathematically complete and physically satisfactory theory, is rather recent. It is only by exploiting the deeper features that one can hope to understand the basic structure of plasma transport theory. Let me immediately say that in this paper I shall only address the so-called "neoclassical plasma transport theory," which has attained a level of completeness making it amenable to a thermodynamic analysis.[2,3] The field of "anomalous transport," in spite of its great practical importance, is still far from having reached a comparable level of completeness, because the underlying physics of plasma instabilities and turbulence is still poorly understood.

The paper is organized as follows. In Section 2 I review briefly the subject of neoclassical transport theory of toroidally confined plasmas, addressing myself to non-plasma physicists. Most proofs are skipped: they can be found in the standard literature, or in a monograph on plasma transport theory, which I shall publish soon. Moreover, for simplicity, the transport theory is exposed in its simplest form, the 13-moment approximation. However, all the results quoted here have been extended to higher approximations. The final results are given in a form that may not be familiar to plasma physicists: it is important for exhibiting some nontrivial relationships between the transport coefficients, which do not seem to have noticed earlier. The third section discusses the possibility of incorporating these ideas into the framework of linear nonequilibrium thermodynamics. The concept of entropy production, and more precisely, of average entropy production, plays a leading role in this discussion.

2. REVIEW OF THE NEOCLASSICAL THEORY OF TRANSPORT IN MAGNETICALLY CONFINED PLASMAS

We consider a plasma consisting of electrons (mass m_e, charge $e_e = -e$, density n_e, temperature T_e) and one ion species (mass m_i, charge $e_i = Ze$, density $n_i = Z^{-1} n_e$, temperature T_i), confined by an axisymmetric toroidal magnetic field **B**. We consider here, for the purpose of illustration, only the simplest magnetic field configuration, the so-called "standard model" or large-aspect-ratio model, which, in toroidal coordinates (Fig. 1), is represented by[2]

$$\mathbf{B}(r, \theta) = \mathbb{B}([\eta(r)/q(r)]\, \mathbf{e}_\theta + [1 + \eta(r) \cos\theta]^{-1}\, \mathbf{e}_\zeta) \tag{2.1}$$

where $\eta(r) = r/R_0$ (R_0 being the major radius of the torus), $q(r)$ is the so-called safety factor, and $\mathbb{B}$ is a constant reference magnetic field; $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\zeta$ are unit vectors along the radial, poloidal, and toroidal directions, respectively. It is assumed that $\eta \ll 1$. The magnetic field is everywhere tangent to a *magnetic surface* belonging to a set of concentric circular tori. Any quantity depending only on the radial coordinate (hence constant on each magnetic surface) is called a *surface quantity*. With any arbitrary function $A(r, \theta, \zeta)$ one can associate a surface quantity $a(r)$, called its *surface average*:

$$a(r) \equiv \langle A(r) \rangle \tag{2.2}$$

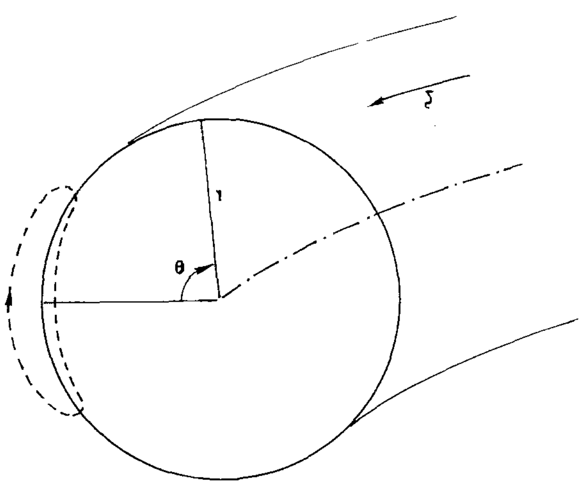

Fig. 1. Toroidal coordinate system. The dashed line is a typical banana orbit.

This averaging operation has the important effect of annihilating the action of the operator $\mathbf{B}\cdot\nabla$ on any scalar quantity:

$$\langle \mathbf{B}(r,\theta)\cdot\nabla f(r,\theta,\zeta)\rangle = 0 \tag{2.3}$$

In the standard model, the averaging operation has the following simple expression:

$$\langle A(r,\theta,\zeta)\rangle = (4\pi^2)^{-1}\int_0^{2\pi} d\theta\, d\zeta\,[1+\eta(r)\cos\theta]\,A(r,\theta,\zeta) \tag{2.4}$$

The *transport phenomena* in such a plasma are characterized by the following *vector fluxes*, which, for convenience, are written in dimensionless form ($\alpha = e, i$):

$$\begin{aligned}
&h_m^{(1)} = (en_e)^{-1}\,(m_e/T_e)^{1/2}\,j_m: \qquad \text{electric current}\\
&h_m^{e(1)} = n_e^{-1}(m_e/T_e)^{1/2}\,\Gamma_m^e: \qquad \text{electron flux}\\
&h_m^{\alpha(3)} = (2/5)^{1/2}\,(m_\alpha/n_\alpha)^{-1}\,(m_\alpha/T_\alpha)^{3/2}\,q_m^\alpha:\\
&\qquad\qquad \text{electron (ion) heat flux}
\end{aligned} \tag{2.5}$$

Associated with these are a set of thermodynamic forces, which are, essentially, the electric field, the pressure gradient, and the electron (ion) temperature gradient. For convenience, their exact form will be given later.

The main purpose of transport theory is the establishment of relations between fluxes and forces. In the case of plasma physics, two regimes have been identified in which these relations are linear: the short-mean-free-path (SMFP) regime [mean free path very small compared to the hydrodynamic length] and the *drift approximation* (Larmor radius very small compared to the hydrodynamic length). The former case corresponds to the "classical" collision-dominated situation, but the latter is much subtler and is characteristic of confined plasma physics. It allows, in particular, a study of long-mean-free-path (LMFP) regimes (which can indeed be realized in present-day high-temperature tokamaks).

Experiments on confined plasmas have no access to the detailed fluxes. Only *surface-averaged fluxes of particles and heat in the radial direction* ("leaks") are measurable. Therefore, the typical transport equations of confined plasma physics are relations between *average* radial fluxes and *average* radial forces: they therefore have a markedly nonlocal character.

Nevertheless, the average components of the fluxes *parallel* to the magnetic field (thus, lying on the magnetic surfaces) also play an important role in the theory. They obey a set of transport equations decoupled from the perpendicular components:

$$\begin{aligned}
\hat{h}_{\|}^{(1)} &= \sigma \hat{g}_{\|}^{(1)A} + (-\sigma B_0^{-1} \langle B \bar{g}_{\|}^{e(1)} \rangle + \alpha B_0^{-1} \langle B \bar{g}_{\|}^{e(3)} \rangle) \\
\hat{h}_{\|}^{e(3)} &= \alpha \hat{g}_{\|}^{(1)A} + (-\alpha B_0^{-1} \langle B \bar{g}_{\|}^{e(1)} \rangle + \kappa^e B_0^{-1} \langle B \bar{g}_{\|}^{e(3)} \rangle) \\
\hat{h}_{\|}^{i(3)} &= \kappa^i B_0^{-1} \langle B \bar{g}_{\|}^{i(3)} \rangle
\end{aligned} \tag{2.6}$$

The average fluxes are defined as follows:

$$\hat{h}_{\|}^{\alpha(P)} = B_0^{-1} \langle \mathbf{B} \cdot \mathbf{h}^{\alpha(P)} \rangle \equiv B_0^{-1} \langle B h_{\|}^{\alpha(P)} \rangle \tag{2.7}$$

Note that the usual parallel thermodynamic forces, which are *gradients* of pressure, temperature, or potential, do not enter these equations: they are annihilated by the surface averaging because of (2.3). Only the nonpotential part of the electric field, $\mathbf{E}^{(A)} = -c^{-1}\, \partial_t \mathbf{A}$, contributes to this equation:

$$\hat{g}_{\|}^{(1)A} = -(m_e/T_e)^{1/2}\, \tau_e (e/m_e)\, B_0^{-1} \langle B E_{\|}^{(A)} \rangle \tag{2.8}$$

where τ_α is the collisional relaxation time of species α. Here B_0 is an average magnetic field:

$$B_0 = \langle B^2 \rangle^{1/2} \tag{2.9}$$

σ, α, and κ^α are the familiar "classical" *parallel* transport coefficients: electrical conductivity, thermoelectric coefficient and thermal conductivities, respectively. In the 13-moment approximation (to which we limit ourselves here) the matrix of the parallel transport coefficients is simply the inverse of the collision operator matrix c_{pq}^α (i.e., c_{pq}^α are the matrix elements of the collision operator for species α in the Hermite representation). In particular,

$$\kappa^i = (c_{33}^i)^{-1} \tag{2.10}$$

The most peculiar quantities entering (2.6) are the *generalized stresses*:

$$\langle B \bar{g}_{\|}^{e(1)} \rangle = -(m_e/T_e)^{1/2}\, (\tau_e/m_e n_e) \langle (\mathbf{B} \cdot \boldsymbol{\nabla})\, \pi^e \rangle \tag{2.11}$$

where π^e is the electron (second-order) dissipative pressure tensor. A similar, somewhat more complicated definition relates $\bar{g}_{\|}^{\alpha(3)}$ to the divergence of the fourth-order stress tensor. These terms, which express the anisotropy of the plasma, play a leading role in the LMFP regime. A fundamental question of transport theory will be the search for their relation to the thermodynamic forces.

Turning now to the *average radial fluxes*, a nontrivial manipulation of the moment equations[2,3] leads to a fundamental decomposition into three contributions:

$$\langle h_r^{\alpha(P)} \rangle = \langle h_r^{\alpha(P)} \rangle_{\rm CL} + \langle h_r^{\alpha(P)} \rangle_{\rm PS} + \langle h_r^{\alpha(P)} \rangle_{\rm B} \tag{2.12}$$

The first term is called the *classical* average radial flux: it is the one that can be obtained from a standard (Chapman–Enskog or Grad type) transport theory[4]:

$$\begin{aligned}
\langle h_r^{e(1)} \rangle_{\rm CL} &= D_{\rm CL}\, g_r^{e(1)} - \alpha_{\rm CL}\, g_r^{e(3)} \\
\langle h_r^{e(3)} \rangle_{\rm CL} &= -\alpha_{\rm CL}\, g_r^{e(1)} + \kappa_{\rm CL}^e\, g_r^{e(3)} \\
\langle h_r^{i(3)} \rangle_{\rm CL} &= \kappa_{\rm CL}^i\, g_r^{i(3)}
\end{aligned} \tag{2.13}$$

where the (dimensionless) *thermodynamic forces* are, respectively, proportional to the radial pressure gradient and temperature gradients:

$$\begin{aligned}
g_r^{e(1)} &= -\tau_e (m_e/T_e)^{1/2}\, (m_e n_e)^{-1}\, \nabla_r P \\
g_r^{e(3)} &= -(5/2)^{1/2}\, \tau_\alpha (T_\alpha/m_\alpha)^{1/2}\, T_\alpha^{-1}\, \nabla_r T_\alpha
\end{aligned} \tag{2.14}$$

These quantities are, to leading order, surface quantities; hence, they do not require any averaging.

The classical transport coefficients appearing here [diffusion coefficient $D_{\rm CL}$, thermodiffusion coefficient $(-\alpha_{\rm CL})$, and thermal conductivities $\kappa_{\rm CL}^\alpha$] are the surface averages of the well-known perpendicular transport coefficients. For instance, the ion thermal conductivity perpendicular to the magnetic field is given by

$$\kappa_\perp^i = c_{33}^i (\Omega_i \tau_i)^{-2} \tag{2.15}$$

where Ω_α is the Larmor frequency for species α:

$$\Omega_\alpha = e_\alpha B(r, \theta)/m_\alpha c \tag{2.16}$$

The "classical" radial ion thermal conductivity is then defined as

$$\kappa_{\rm CL}^i = \langle \kappa_\perp^i \rangle = c_{33}^i (\Omega_{i0} \tau_i)^{-2}\, \mathscr{G} \tag{2.17}$$

where

$$\mathscr{G} = \langle B_0^2/B^2 \rangle \tag{2.18}$$

and $\Omega_{\alpha 0}$ is the Larmor radius evaluated with the average magnetic field B_0, a surface quantity.

The second term in Eq. (2.12) is called the *Pfirsch–Schlüter* (PS) radial average flux. These authors[5] discovered that an inhomogeneity of the pressure or temperature on the magnetic surface can influence the transport in the perpendicular radial direction. This purely geometrical effect can be interpreted in terms of drift motions of the charged particles. It was later derived in an elegant way from kinetic theory[2,6]; a further improvement demonstrates the universality of the result. This discovery came at as a great surprise, because it showed that a purely geometrical cause can have a (big!) effect on the transport phenomena. The Pfirsch–Schlüter fluxes can be related to the thermodynamic forces by the *same* equations (2.13), in which one simply changes the values of all the classical coefficients by a "PS amplification factor":

$$L_{\mathrm{CL}} \to L_{\mathrm{PS}} = [(q/\eta)^2(\mathscr{G}-1)/\mathscr{G}]\, L_{\mathrm{CL}} \tag{2.19}$$

Thus, for instance, the PS ion thermal conductivity is obtained from (2.17) as follows:

$$\kappa^i_{\mathrm{PS}} = \mathscr{F}_i^2(\mathscr{G}-1)\, c^i_{33} \tag{2.20}$$

where I introduced the important quantity

$$\mathscr{F}_\alpha = q/\eta\Omega_{\alpha 0}\tau_\alpha \tag{2.21}$$

Equation (2.19) clearly shows the geometrical origin of the PS effect: the latter disappears in a straight (or, *a fortiori*, in a homogeneous) magnetic field, where $\mathscr{G}=1$. It can be shown that the PS transport coefficient dominates the classical one by a factor $2q^2 \approx 10$, but we do not wish to discuss this very well-known fact here.

Finally, the third term in (2.12) is called the *banana* average radial flux and is directly related to the generalized stresses:

$$\langle h_r^{\alpha(n)}\rangle_{\mathrm{B}} = \mathscr{F}_\alpha B_0^{-1}\langle B\tilde{g}_{\|}^{\alpha(n)}\rangle \tag{2.22}$$

The strange name comes from the fact that in a long-mean-free-path situation there exists a class of particles that are trapped in the local magnetic mirrors intrinsically present in a toroidal configuration. These particles may perform many oscillations along an orbit having the shape of a banana (see Fig. 1) before suffering a collision. It turns out that the trapped particles are primarily responsible for these contributions to the fluxes. The banana fluxes were discovered by Galeev and Sagdeev[7]; their theory was refined by Hazeltine *et al.*[8] and later improved by Hirshman and Sigmar.[3]

In order to evaluate the generalized stresses, one first exploits another purely geometrical effect: it is shown that, to leading order in the drift approximation, all the vector fluxes must have *zero divergence.* A nontrivial utilization of this condition leads to a relation between *parallel* average fluxes and *radial* thermodynamic forces:

$$\hat{h}_{\|}^{\alpha(n)} = \mathscr{F}_\alpha g_r^{\alpha(n)} + \omega_n^\alpha, \qquad n = 1, 3; \quad \alpha = e, i \tag{2.23}$$

The quantities ω_n^α, called (rather improperly) "poloidal fluxes," have the important property of being surface quantities: they are not determined by the zero-divergence condition (they appear as integration constants in the solution of a "magnetic differential equation"). In order to fix their values, one identifies the right-hand sides of (2.23) and (2.6), thus obtaining a relation between generalized stresses, poloidal fluxes, and thermodynamic forces. An additional set of relations is obtained by a direct solution of the kinetic equation in the LMFP regime. This is the truly nontrivial part of the calculation, into which I cannot enter here. I simply quote the remarkable end result,[3] which is a linear relation between generalized stresses and poloidal fluxes:

$$B_0^{-1}\langle B\bar{g}_{\|}^{\alpha(n)}\rangle = -\varphi[\mu_{n1}^\alpha \omega_1^\alpha + \mu_{n3}^\alpha \omega_3^\alpha], \qquad n = 1, 3 \tag{2.24}$$

where φ is the important "neoclassical factor"

$$\varphi = \frac{1 - f_p}{f_p}, \qquad f_p = \frac{3}{4} B_0^2 \int_0^{\lambda_c} d\lambda \frac{\lambda}{\langle (1 - \lambda B)^{1/2} \rangle} \tag{2.25}$$

φ is sometimes loosely called the ratio of trapped to untrapped particles. The coefficients μ_{nm}^α are called "*pseudo-viscosity coefficients*" and are determined only by the collision term: their numerical values are easily calculated. It should be noted that they form a symmetric matrix: $\mu_{nm}^\alpha = \mu_{mn}^\alpha$. Equations (2.24) provide the last missing link in the theory. They are now combined with (2.6) and (2.22) and lead to a final set of linear banana transport equations, which, for convenience, are written in the following form:

$$\begin{aligned}
\langle h_r^{e(p)} \rangle_{\mathrm{B}} &= \mathscr{F}_e^2 \varphi [l_{p1}^{ee}(\varphi)\, g_r^{e(1)} + l_{p31}^{ee}(\varphi)\, g_r^{e(3)} + l_{p3}^{ei}(\varphi)\, aAg_r^{i(3)}] \\
&\quad + \mathscr{F}_\alpha \varphi l_{pE}^e(\varphi)\, \hat{g}_{\|}^{1(A)}, \qquad p = 1, 3 \\
\langle h_r^{i(3)} \rangle_{\mathrm{B}} &= \mathscr{F}_e^2 \varphi a^{-1} [l_{31}^{ie}(\varphi)\, g_r^{e(1)} + l_{33}^{ie}(\varphi)\, g_r^{e(3)}] \\
&\quad + \mathscr{F}_i^2 \varphi l_{33}^{ii}(\varphi)\, g_r^{i(3)} + \mathscr{F}_\alpha \varphi a^{-1} l_{3E}^i(\varphi)\, \hat{g}_{\|}^{1(A)}
\end{aligned} \tag{2.26a}$$

As the average radial fluxes appear to be coupled to the parallel electric field $\hat{g}_{\|}^{1(A)}$, these equations must be completed by a relation for the flux conjugate to the latter, i.e., the banana contribution to the average parallel electric current; it is obtained from the last two terms on the right-hand side of the first equation (2.6):

$$\hat{h}^{(1)}_{\|B} = \mathscr{F}_e \varphi [l^e_{E1}(\varphi)\, g^{e(1)}_r + l^e_{E3}(\varphi)\, g^{e(3)}_r + l^i_{E3}(\varphi)\, g^{i(3)}_r] + \varphi l^e_{E1}(\varphi)\, \hat{g}^{1(A)}_{\|} \qquad (2.26b)$$

I used the following abbreviations:

$$a = \left(\frac{T_i}{T_e}\frac{m_e}{m_i}\right)^{1/2}, \qquad A = \frac{|\Omega_{e0}|\,\tau_e}{\Omega_{i0}\tau_i}$$

Usually A is a large number: the banana transport coefficients are conveniently evaluated in the limit $A \gg 1$.

I now discuss these transport equations in some detail. The first unusual feature is the coupling of the radial fluxes to *all* the thermodynamic forces. In particular, the *electronic* fluxes are connected to the *ionic* forces (and conversely); moreover, the *radial* fluxes are connected to a *parallel* electric field. These features are in sharp contrast to the classical and PS structure of Eq. (2.13).

A deeper study of the banana transport coefficients (which does not seem to have been previously done) reveals some striking structural features. It turns out that these coefficients can be divided into three categories. The first one includes the "*pure diffusive coefficients*" l^{ee}_{11}, l^{ee}_{13}, l^{ee}_{31}, l^{ee}_{33}, l^{ii}_{33}. They form a matrix of the same structure as the classical one (2.13), and which has the same fundamental "thermodynamic property" of being a symmetric, positive-definite matrix:

$$l^{ee}_{13} = l^{ee}_{31} \qquad (2.27a)$$

$$l^{\alpha\alpha}_{pp} > 0, \qquad l^{ee}_{11} l^{ee}_{33} - (l^{ee}_{13})^2 > 0 \qquad (2.27b)$$

These coefficients are combinations of the classical parallel transport coefficients [i.e., the coefficients σ, α, κ^α of Eq. (2.6)] and the pseudo-viscosity coefficients of Eq. (2.24). We do not list them here, but give a typical example:

$$l^{ee}_{11}(\varphi) = D_e^{-1}(\varphi)\{\mu^e_{11} + \varphi\kappa^e[\mu^e_{11}\mu^e_{33} - (\mu^e_{13})^2]\}$$

with

$$\begin{aligned} D_e(\varphi) = 1 &+ \varphi(\sigma\mu^e_{11} - 2\alpha\mu^e_{13} + \kappa^e\mu^e_{33}) \\ &+ \varphi^2(\sigma\kappa^e - \alpha^2)[\mu^e_{11}\mu^e_{33} - (\mu^e_{13})^2] \end{aligned}$$

The second class of transport coefficients includes the "*mixed diffusive coefficients*" coupling the electrons to the ions. It turns out that these are not independent of the former:

$$l^{ei}_{p3}(\varphi) = m_i(\varphi)\, l^{ee}_{p1}(\varphi), \qquad p = 1, 3 \tag{2.28}$$

with

$$m_i(\varphi) = \mu^i_{13}\{\mu^i_{11} + \varphi\kappa^i[\mu^i_{11}\mu^i_{33} - (\mu^i_{13})^2]\}^{-1}$$

The third class comprises the "*electrical coefficients*": these also depend on the basic diffusive coefficients:

$$\begin{aligned} l^{\alpha}_{pE}(\varphi) &= \sigma l^{\alpha e}_{p1}(\varphi) - \alpha l^{\alpha e}_{p3}(\varphi) \\ l_{EE}(\varphi) &= -\sigma l^{e}_{1E}(\varphi) + \alpha l^{e}_{3E}(\varphi) \end{aligned} \tag{2.29}$$

The global banana transport matrix has the following symmetry properties:

$$\begin{aligned} l^{\alpha\beta}_{pq}(\varphi) &= l^{\beta\alpha}_{qp}(\varphi) \qquad && p, q = 1, 3 \\ l^{\alpha}_{pE}(\varphi) &= -l^{\alpha}_{Ep}(\varphi), && p = 1, 3 \end{aligned} \tag{2.30}$$

This is in agreement with Onsager's principle (the antisymmetry of the electrical cross-coefficients is due to their being odd functions of the magnetic field). A surprise comes, however, from the property

$$l_{EE}(\varphi) < 0 \tag{2.31}$$

There are good physical reasons for this behavior: the trapped particles cannot participate in a parallel current, hence the parallel electrical conductivity is diminished in their presence; moreover, the "total" electrical conductivity $\sigma + \mathscr{F}^2_e \varphi l^e_{EE}(\varphi)$ [see Eq. (2.6)] remains positive for all values of φ.

3. ENTROPY PRODUCTION AND TRANSPORT PROCESSES IN MAGNETICALLY CONFINED PLASMAS

At the end of the review of the neoclassical transport theory one is left with a vague feeling of uneasiness. The average radial fluxes involve three contributions (2.12), two of which obey transport equations of the usual type of the "phenomenological equations" of irreversible thermodynamics, but the third does not. It may be argued that the banana fluxes are characteristic of the long-mean-free-path regime, to which thermodynamics does not apply. But this argument, as such, is weak, because the constituents of the banana transport coefficients, namely the parallel transport coefficients and the pseudoviscosities, are both determined by the collision operator of the kinetic equation and should therefore reflect its properties. It is therefore important, for understanding the status of this theory, to investigate whether (or how) it enters the framework of thermodynamics.

The natural starting point for this study is the concept of *entropy production*. Kinetic theory provides us with a quite general expression of this quantity. When the latter is evaluated with a distribution function expressed as a superposition of Hermitian moments and is then truncated by retaining only leading terms (quadratic in the fluxes) and is moreover limited to the "13-moment" approximtion, the following form is found for the density of entropy production σ^α (made dimensionless by the factor τ_α/n_α):

$$\begin{aligned}\Sigma^e &\equiv (\tau_e/n_e)\,\sigma^e = c^e_{11}\mathbf{h}^{(1)}\cdot\mathbf{h}^{(1)} + c^e_{33}\mathbf{h}^{e(3)}\cdot\mathbf{h}^{e(3)} + 2c^e_{13}\mathbf{h}^{(1)}\cdot\mathbf{h}^{e(3)}\\ \Sigma^i &\equiv (\tau_i/n_i)\,\sigma^i = c^i_{33}\mathbf{h}^{i(3)}\cdot\mathbf{h}^{i(3)}\end{aligned} \tag{3.1}$$

where c^α_{pq} are the matrix elements of the collision operator. I stress the fact that this "kinetic form" of the entropy production appears as a quantity intrinsically characteristic of the collision operator: it is simply the quadratic form associated with the symmetric collision matrix. In particular, no assumption about the size of the mean free path has been made here.

Making use of the moment equations, one can transform (3.1) into

$$\Sigma^i = \mathbf{h}^{i(3)}\cdot\mathbf{g}^{i(3)} + \mathbf{h}^{i(3)}\cdot\bar{\mathbf{g}}^{i(3)} \tag{3.2}$$

For brevity, I henceforth only write out the shorter formulas for the ions, it being understood that similar expressions hold for the electronic entropy production. If the generalized stresses $\bar{\mathbf{g}}^{i(3)}$ were zero (as in a short-mean-free-path regime), (3.3) would reduce to the usual thermodynamic bilinear form of fluxes h and forces g. Here, however, we have a "quasi-thermodynamic form."

I now make use of the transport equations relating the fluxes to the forces and eliminate some terms that are negligible in the drift approximation[2]; then (3.2) reduces to

$$\begin{aligned}\Sigma^i &= \kappa^i(g^{i(3)}_{\|} + \bar{g}^{i(3)}_{\|})^2 + \kappa^i_\perp(g^{i(3)}_r)^2\\ &\equiv \Sigma^i_{\|} + \Sigma^i_{\mathrm{CL}}\end{aligned} \tag{3.3}$$

[2] These are the "perp-tangential" components of the vectors (perpendicular to the magnetic field and to the radial direction) as well as the contribution of the radial generalized stresses.

The second term, denoted Σ^i_{CL}, has the "ordinary" thermodynamic form of a definite positive quadratic form in the forces, whose coefficients are the perpendicular (unaveraged!) transport coefficients of Eq. (2.15). On the contrary, the parallel part (which involves the parallel heat conductivity, a surface quantity), contains a contribution of the generalized stresses and is therefore *not* in the standard form. Clearly, the neoclassical effects are hidden in the parallel entropy production.

According to the general philosophy of confined plasma physics, it should be clear that only surface-averaged quantities are expected to be physically relevant. This statement should apply to the entropy production as well. We therefore define a *surface-averaged entropy production*:

$$\langle \Sigma^i \rangle = \langle \Sigma^i \rangle_{\parallel} + \langle \Sigma^i \rangle_{\mathrm{CL}} \tag{3.4}$$

The *classical contribution* poses no problem. Indeed, in Eq. (3.3), $g_r^{i(3)}$ is a surface quantity; hence the average only involves the perpendicular heat conductivity:

$$\langle \Sigma^i \rangle_{\mathrm{CL}} = \langle \kappa^i_{\perp} \rangle (g_r^{i(3)})^2 \tag{3.5}$$

and we know from (2.17) that $\langle \kappa^i_{\perp} \rangle = \kappa^i_{\mathrm{CL}}$. Thus, the classical average entropy production has the canonical "transport form" (3.3), i.e., a quadratic form in the forces, whose coefficients are the averaged classical radial transport coefficients. This property also implies the *factorization* inherent to the thermodynamic form (3.2):

$$\langle \Sigma^i \rangle_{\mathrm{CL}} = \langle h_r^{i(3)} \rangle_{\mathrm{CL}} \, g_r^{i(3)} \tag{3.6}$$

i.e., a bilinear form in the *average* fluxes and forces, in agreement with standard nonequilibrium thermodynamics.

For the *parallel* average entropy production, the situation is not so clear. Indeed, using the simpler kinetic form (3.1), we have

$$\langle \Sigma^i \rangle_{\parallel} = c^i_{33} \langle h_{\parallel}^{i(3)} h_{\parallel}^{i(3)} \rangle \tag{3.7}$$

and this is *not* a quadratic form.

In order to clarify the situation, we note that a (nontrivial) manipulation of the moment equations and of the zero-divergence constraint leads to a remarkable expression of the (unaveraged!) parallel fluxes:

$$h_{\parallel}^{i(3)} = \left(\frac{B_0}{B} - \frac{B}{B_0} \right) \mathscr{F}_i \, g_r^{i(3)} + \frac{B}{B_0} \kappa^i \left\langle \frac{B}{B_0} \, \bar{g}_{\parallel}^{i(3)} \right\rangle \tag{3.8}$$

[Recalling that the dependence on the poloidal angle is contained solely in $B(r, \theta)$, Eq. (2.1), it is easily checked that this equation, upon multiplication by (B/B_0) and surface averaging, reduces to (2.6)]. But through Eq. (2.22), this equation is transformed into a relation between parallel fluxes and average radial banana fluxes:

$$\begin{aligned} h_{\|}^{i(3)} &= \left(\frac{B_0}{B} - \frac{B}{B_0}\right) \mathscr{F}_i g_r^{i(3)} + \frac{B}{B_0} \kappa^i \mathscr{F}_i^{-1} \langle h_r^{i(3)} \rangle_{\mathrm{B}} \\ &\equiv h_{\|\mathrm{PS}}^{i(3)} + h_{\|\mathrm{B}}^{i(3)} \end{aligned} \tag{3.9}$$

Such a relation between parallel and radial fluxes and forces is ultimately due to the geometrical constraint of zero divergence, mentioned in Section 2. (The reason for the choice of indices in the abbreviated notation will presently become clear.)

The main interest of (3.9) is that the poloidal angle dependence [through $B(\theta, r)$] is explicitly exhibited. Therefore, after substitution into (3.7), the surface average can be calculated exactly. We note the following obvious identities [recalling (2.9)]:

$$\left\langle \frac{B^2}{B_0^2} \right\rangle = 1$$

$$\left\langle \left(\frac{B_0}{B} - \frac{B}{B_0}\right)^2 \right\rangle = \mathscr{G} - 1$$

$$\left\langle \frac{B}{B_0} \left(\frac{B_0}{B} - \frac{B}{B_0}\right) \right\rangle = 0$$

Hence, the average entropy production splits into two terms:

$$\langle \Sigma^i \rangle_{\|} = \langle \Sigma^i \rangle_{\mathrm{PS}} + \langle \Sigma^i \rangle_{\mathrm{B}} \tag{3.10}$$

$$\langle \Sigma^i \rangle_{\mathrm{PS}} = \mathscr{F}_i^2 (\mathscr{G} - 1)\, c_{33}^i (g_r^{i(3)})^2 \tag{3.11}$$

$$\langle \Sigma^i \rangle_{\mathrm{B}} = \mathscr{F}_i^{-2} \kappa^i \langle h_r^{i(3)} \rangle_{\mathrm{B}}^2 \tag{3.12}$$

The first remarkable fact in this relation is the following: the average of a product is represented as a *sum of two products* of forces or fluxes. It is this factorization theorem that enables us to provide a thermodynamic interpretation of the neoclassical results.

Next, we note that the coefficient in (3.11) is simply the *Pfirsch–Schlüter radial transport coefficents* κ_{PS}^i defined in (2.20). Thus, we have isolated a Pfirsch–Schlüter and a banana contribution to the average entropy production. The former has exactly the canonical form (3.5) of a quadratic form in the forces, associated with the positive-definite PS transport matrix. In other words, the PS average entropy production has the standard thermodynamic form:

$$\langle \Sigma^i \rangle_{\mathrm{PS}} = \langle h_r^{i(3)} \rangle_{\mathrm{PS}}\, g_r^{i(3)} \tag{3.13}$$

It can be shown that this relation is quite general: in particular, it is independent of the truncation at the 13-moment level of the transport theory.

Although we found such a nice structure, we are still puzzled by the following question. If the PS fluxes enter so naturally into the thermodynamic framework, why should there be a PS effect *in addition* to the classical fluxes after all? The entropy production provides us with a quite original answer to this question.

Combining Eqs. (2.6) and (2.21), we find the following relation between the (total) *average* parallel heat flux and the average radial banana flux:

$$\hat{h}_{\parallel}^{i(3)} = \mathscr{F}_i^{-1} \kappa^i \langle h_r^{i(3)} \rangle_{\mathrm{B}} \tag{3.14}$$

Combining now Eqs. (3.7), (3.10)–(3.12), and (3.14), we find

$$\langle \Sigma^i \rangle_{\parallel} - \langle \Sigma^i \rangle_{\mathrm{B}} = c_{33}^i \langle h_{\parallel}^{i(3)} h_{\parallel}^{i(3)} \rangle - (\kappa^i)^{-1}\, \hat{h}_{\parallel}^{i(3)} \hat{h}_{\parallel}^{i(3)} = \langle \Sigma^i \rangle_{\mathrm{PS}}$$

or, using (2.10),

$$\langle h_{\parallel}^{i(3)} h_{\parallel}^{i(3)} \rangle - \hat{h}_{\parallel}^{i(3)} \hat{h}_{\parallel}^{i(3)} = \kappa^i \langle \Sigma^i \rangle_{\mathrm{PS}} = (\mathscr{G} - 1)\, \mathscr{F}_i^2 (g_r^{i(3)})^2 \tag{3.15}$$

Thus, *the PS effect originates from the spatial fluctuations of the parallel fluxes on a magnetic surface. The PS average entropy production is precisely a measure of these fluctuations.* Their explicit value is proportional to the square of the radial forces, multiplied by the geometrical factor $(\mathscr{G} - 1)\, \mathscr{F}_i^2$.

We now turn to the *banana average entropy production.* Equation (3.12) is queer, for the following reason: it expresses the average entropy production as a standard quadratic form associated with the *parallel* transport matrix, but the variables involved are *not* thermodynamic *forces* as would be expected [see (3.5)], but rather the average *radial* banana *fluxes.* On second thought, however, the situation is not unreasonable: indeed, because of (2.22), Eq. (3.12) can be rewritten as

$$\langle \Sigma^i \rangle_{\mathrm{B}} = \kappa^i (B_0^{-1} \langle B \bar{g}_{\parallel}^{i(3)} \rangle)^2 \tag{3.16}$$

This is the standard transport form, in which the thermodynamic forces are replaced by the generalized stresses. It is clear from (3.2), (3.3) that the generalized stresses play a role quite analogous to the true thermodynamic forces.

The average banana entropy production therefore assumes the "*quasithermodynamic form*":

$$\langle \Sigma^i \rangle_{\mathrm{B}} = \hat{h}_{\|}^{i(3)} (B_0^{-1} \langle B \bar{g}_{\|}^{i(3)} \rangle) \tag{3.17}$$

which implies that the "thermodynamic force" conjugate to the average parallel heat flux is the corresponding average generalized stress, as follows from (2.6).

This seems to be the only reasonable extension of nonequilibrium thermodynamic concepts to the banana transport phenomena: it conserves the formal structure of the entropy production as a bilinear form in the forces and the fluxes. When written as a quadratic form, it introduces the *parallel* transport coefficients as the basic transport matrix.

Of course, banana transport theory shows that the generalized stresses can, in turn, be connected to the ordinary thermodynamic forces. It is a trivial matter to substitute the transport equation (2.26a) into (3.12). The result is an expression of the average entropy production as a quadratic form in the true thermodynamic forces (we now write expressions for both species):

$$\langle \Sigma^\alpha \rangle_{\mathrm{B}} = \sum_{p,q=1}^{4} L_{pq}^\alpha X_p X_q, \qquad \alpha = e, i \tag{3.18}$$

where we adopt a more compact notation:

$$X_1 = g_r^{e(1)}, \qquad X_2 = g_r^{e(3)}, \qquad X_3 = g_r^{i(3)}, \qquad X_4 = \hat{g}_{\|}^{(1)A}$$

Clearly, both the electron and the ion banana average entropy production will be expressed in the same form, because the same forces appear in all the transport equations (2.27): this is a difference with the traditional forms of the entropy production, in which ionic and electronic forces are clearly separated.

The coefficients L_{pq}^α appearing in (3.18) are *not* the banana transport coefficients $l_{pq}^{\alpha\beta}$: they are rather complicated combinations of parallel transport coefficients and of banana coefficients. For instance,

$$L_{44}^e = \varphi^2 [\sigma (l_{1E}^e)^2 + \kappa^e (l_{3E}^e)^2 - 2\alpha l_{1E}^e l_{3E}^e - 2\varphi^{-1} l_{EE} + \varphi^{-2} \sigma] > 0$$

I quote this example in order to stress the following fact: the coefficients entering the quadratic forms (3.18) form a *symmetric positive matrix*. This was not the case for the banana transport matrix: recall (2.31). There is, however, a difference between (3.18) and the ordinary entropy production: (3.18) is a *semidefinite* (rather than definite) positive form (i.e., it can vanish for nonzero values of its arguments X_p). This is reflected by the fact that some of the second-order determinants constructed from the L^{α}_{pq} are zero. [This property is trivial for the ion case, but not for the electron case: in the latter, the structural relations (2.28)–(2.29) play an important role in the proof of this property.] This fact expresses the lack of independence of the variables X_p in these forms. In the ion case this is obvious: the average entropy production basically contains a single square term (3.12), which has been "artificially" expanded by writing the banana flux as a combination of four terms. As a result of this discussion, it is clear that the average banana entropy production *cannot* be written in the standard thermodynamic form:

$$\langle \Sigma^i \rangle_{\mathrm{B}} \neq \langle h_r^{i(3)} \rangle_{\mathrm{B}}\, g_r^{i(3)} \tag{3.19}$$

We have found that there appears to be a "divorce" between the ordinary banana transport coefficients entering the transport equations (2.26) and the "entropic" transport coefficients defining the entropy production (3.18). The correspondence between transport theory and thermodynamics is maintained at the level of a description in terms of parallel fluxes and generalized stresses, (3.17). One could therefore argue that the difficulty encountered here is due to the banana fluxes being actually "disguised parallel fluxes." (More precisely, the banana radial fluxes are induced by parallel fluxes and forces through the geometrical zero-divergence constraint.) But this argument cannot explain everything. Indeed, the PS radial fluxes are also "disguised parallel fluxes"; nevertheless, they are well integrated in the framework of thermodynamics.

The answer to the difficulty is probably that the banana fluxes are typical of a long-mean-free-path regime. The distribution functions are markedly different from the short-mean-free-path, strongly collisional regime, which naturally leads to the classical form of the entropy production. I have thus returned to my initial statement, but made it much more precise. If this is so, it is remarkable that banana transport theory can be formulated in terms that, though not identical, are as close as possible to nonequilibrium thermodynamics.

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